GRAPH THEORY

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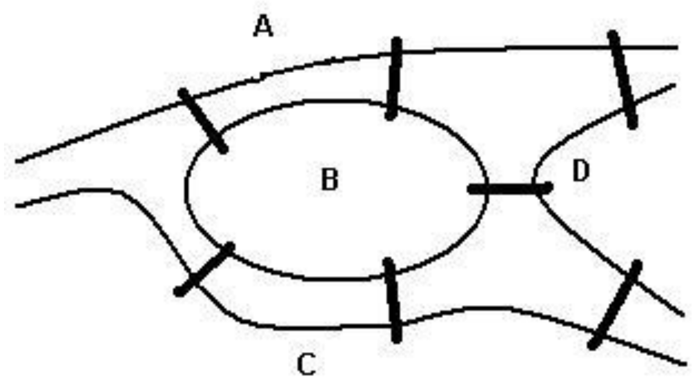
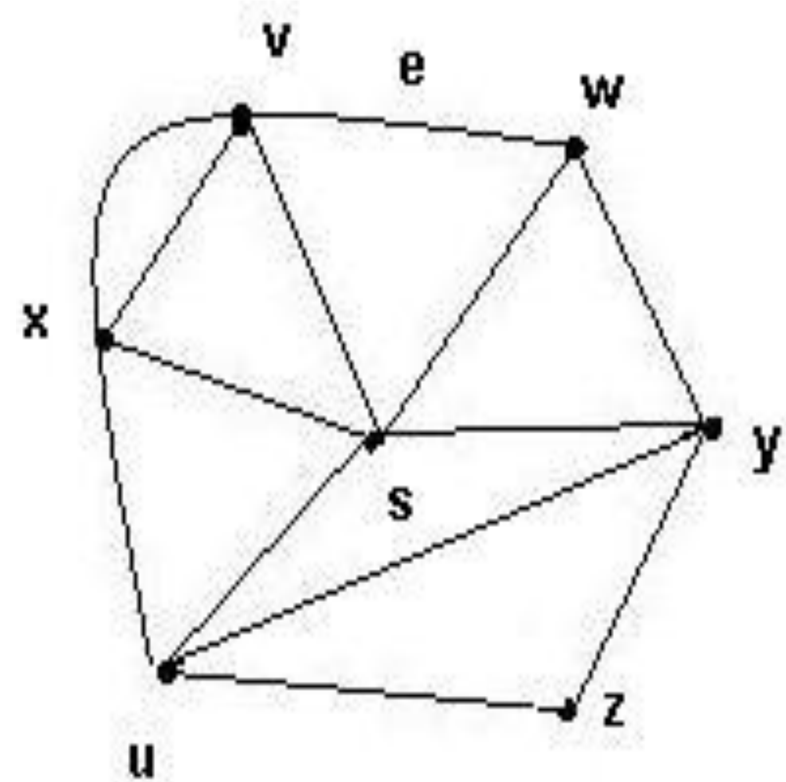
In the beginning…



- 1736: **Leonhard Euler**
  - Basel, 1707-St. Petersburg, 1786
  - He wrote *A solution to a problem concerning the geometry of a place.* First paper in graph theory.
- Problem of the Königsberg bridges:
  - Starting and ending at the same point, is it possible to cross all seven bridges just once and return to the starting point?

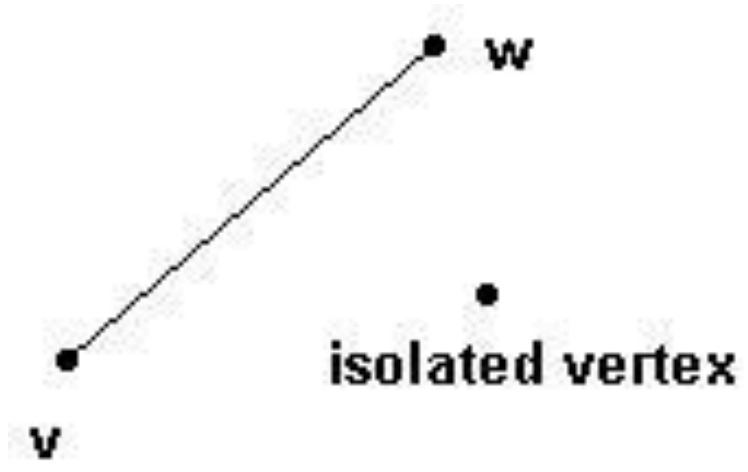
Introduction

- What is a graph G?
- It is a pair G = (V, E), where
  - V = V(G) = set of vertices
  - E = E(G) = set of edges
- **Example:**
  - V = {s, u, v, w, x, y, z}
  - E = {(x,s), $(x,v)_1$, $(x,v)_2$, (x,u), (v,w), (s,v), (s,u), (s,w), (s,y), (w,y), (u,y), (u,z),(y,z)}



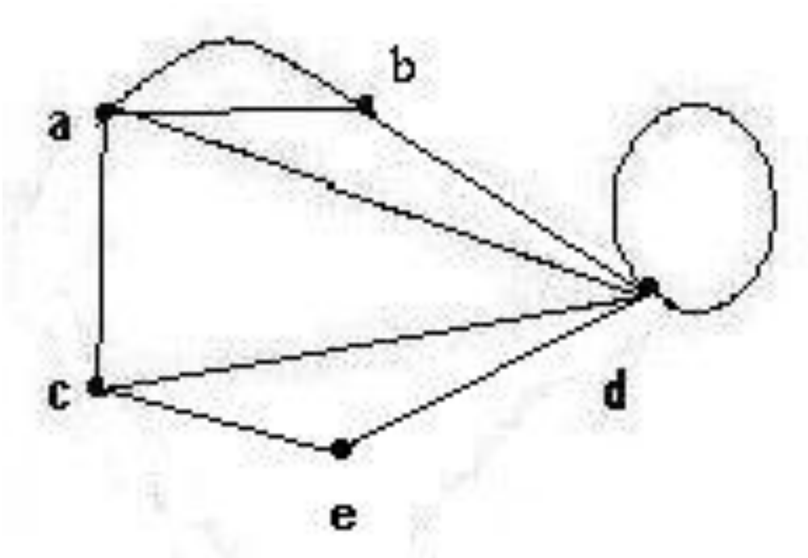
Edges

- An edge may be labeled by a pair of vertices, for instance e = (v,w).
- e is said to be *incident* on v and w.
- Isolated vertex = a vertex without incident edges.



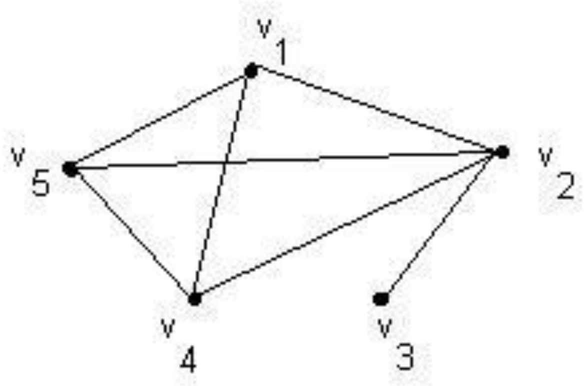
Special edges

- Parallel edges
  - Two or more edges joining a pair of vertices
    - in the example, **a** and **b** are joined by two parallel edges
- Loops
  - An edge that starts and ends at the same vertex
    - In the example, vertex **d** has a loop

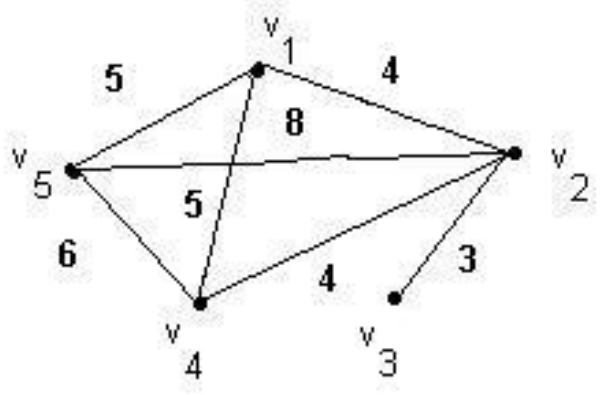


Special graphs

- Simple graph
  - A graph without loops or parallel edges.

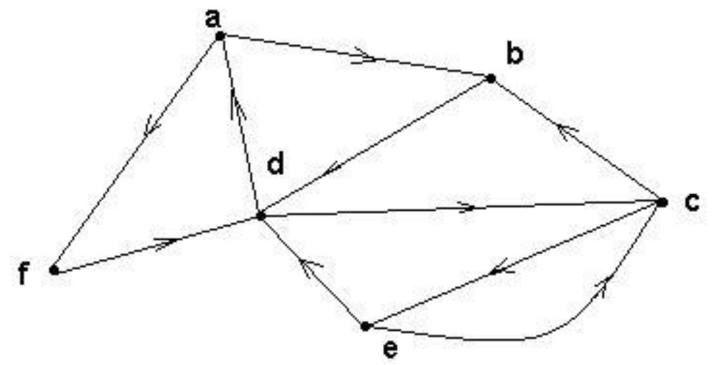


- Weighted graph
  - A graph where each edge is assigned a numerical label or “weight”.



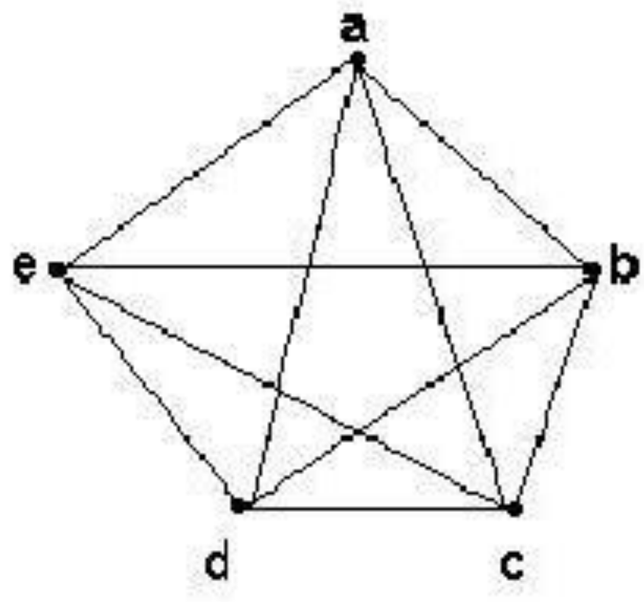
Directed graphs (digraphs)

G is a *directed graph* or *digraph* if each edge has been associated with an ordered pair of vertices, i.e. each edge has a direction



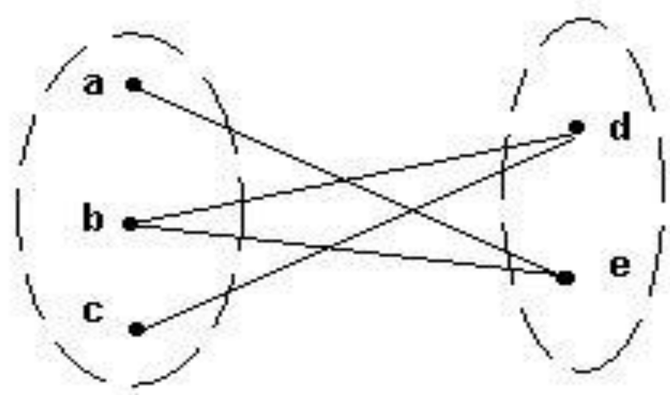
Complete graph K_n

- Let $n \geq 3$
- The *complete graph* K_n is the graph with n vertices and every pair of vertices is joined by an edge.
- The figure represents K_5

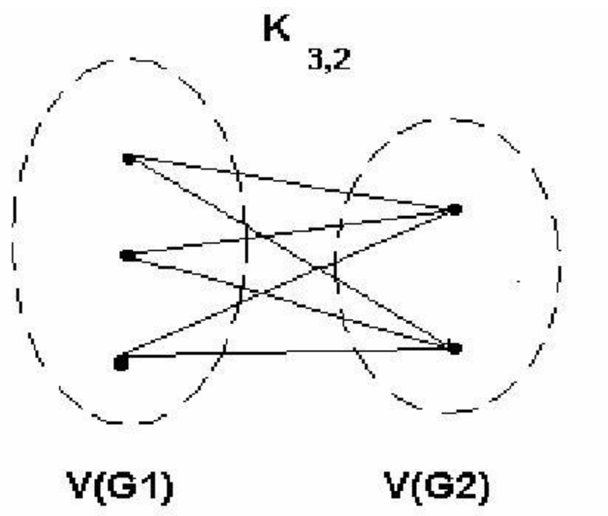


Bipartite graphs

- A bipartite graph G is a graph such that
  - $V(G) = V(G_1) \cup V(G_2)$
  - $|V(G_1)| = m$, $|V(G_2)| = n$
  - $V(G_1) \cap V(G_2) = \varnothing$
  - No edges exist between any two vertices in the same subset $V(G_k)$, $k = 1,2$



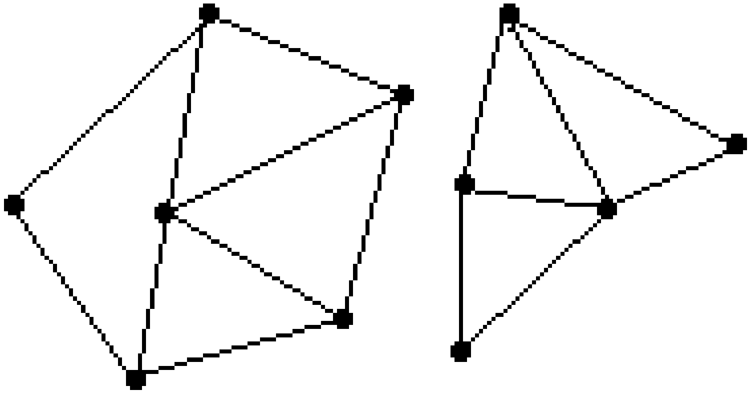
Complete bipartite graph $K_{m,n}$



- A bipartite graph is the *complete* bipartite graph $K_{m,n}$ if every vertex in $V(G_1)$ is joined to a vertex in $V(G_2)$ and conversely,
- $|V(G_1)| = m$
- $|V(G_2)| = n$

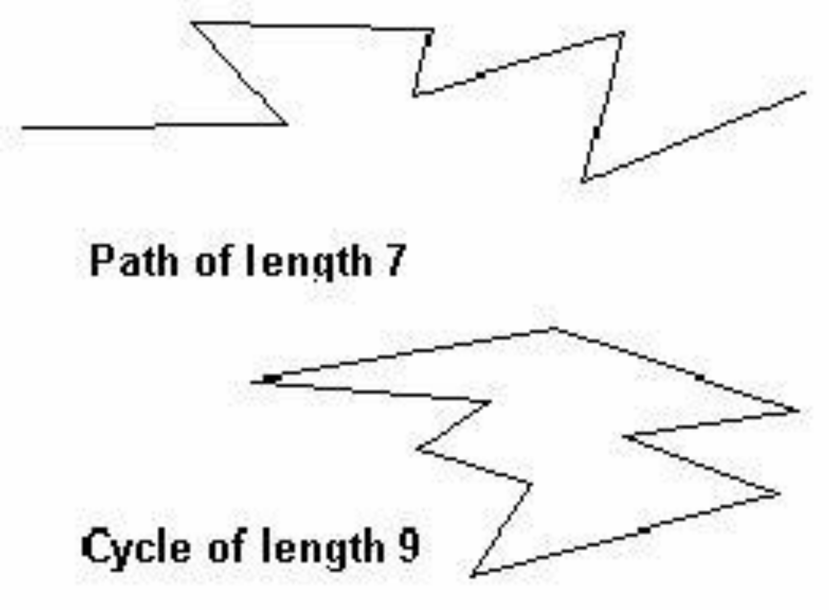
Connected graphs

- A graph is *connected* if every pair of vertices can be connected by a path
- Each connected subgraph of a non-connected graph G is called a *component* of G



2 connected components

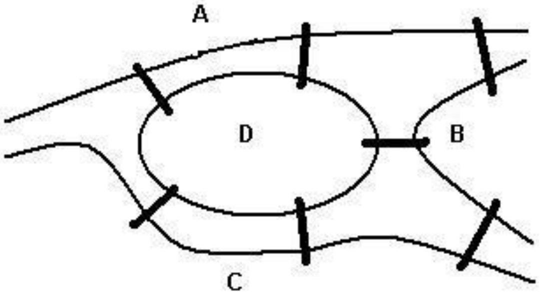
Paths and cycles

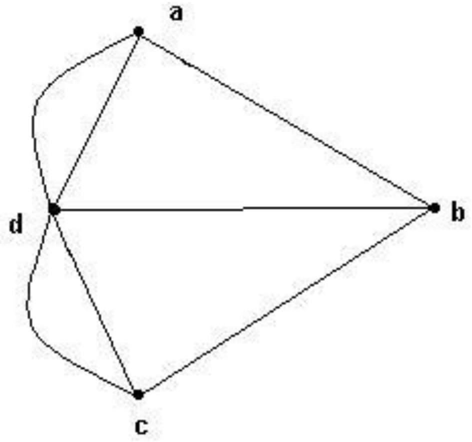


- A *path of length n* is a sequence of n + 1 vertices and n consecutive edges
- A *cycle* is a path that begins and ends at the same vertex

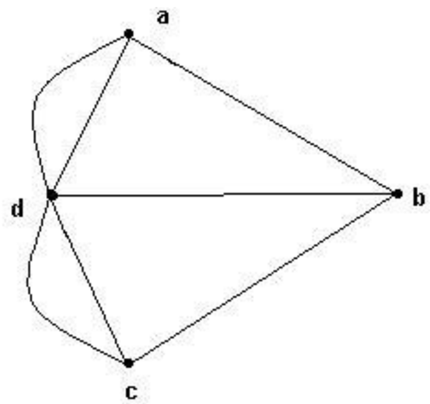
Euler cycles

- An *Euler cycle* in a graph G is a simple cycle that passes through every edge of G only once.
- The Königsberg bridge problem:
  - Starting and ending at the same point, is it possible to cross all seven bridges just once and return to the starting point?
- This problem can be represented by a graph
- Edges represent bridges and each vertex represents a region.





Euler graphs



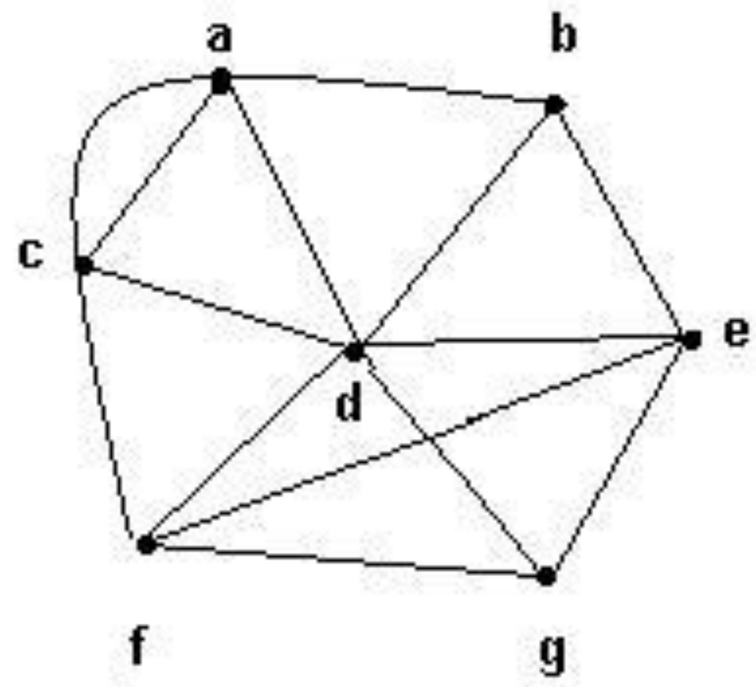
- A graph G is an *Euler graph* if it has an Euler cycle.

Theorem : G is an Euler graph if and only if G is connected and all its vertices have even degree.

- The connected graph represents the Konigsberg bridge problem.
- It is not an Euler graph.
- Therefore, the Konigsberg bridge problem has *no solution*.

Degree of a vertex

- The *degree* of a vertex v, denoted by δ(v), is the number of edges incident on v
- Example:
  - δ(a) = 4, δ(b) = 3,
  - δ(c) = 4, δ(d) = 6,
  - δ(e) = 4, δ(f) = 4,
  - δ(g) = 3.



Sum of the degrees of a graph

Theorem : If G is a graph with m edges and n vertices $v_1, v_2, \ldots, v_n$, then

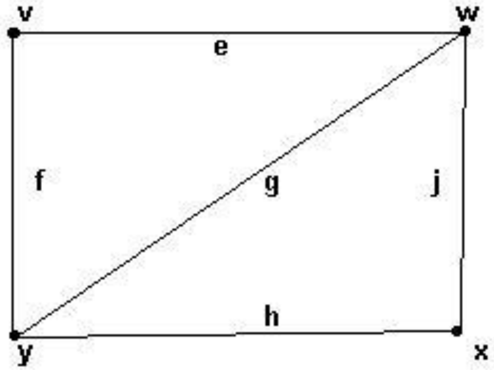
$$\sum_{i=1}^{n} \delta(v_i) = 2m$$

In particular, the sum of the degrees of all the vertices of a graph is even.

Representations of graphs

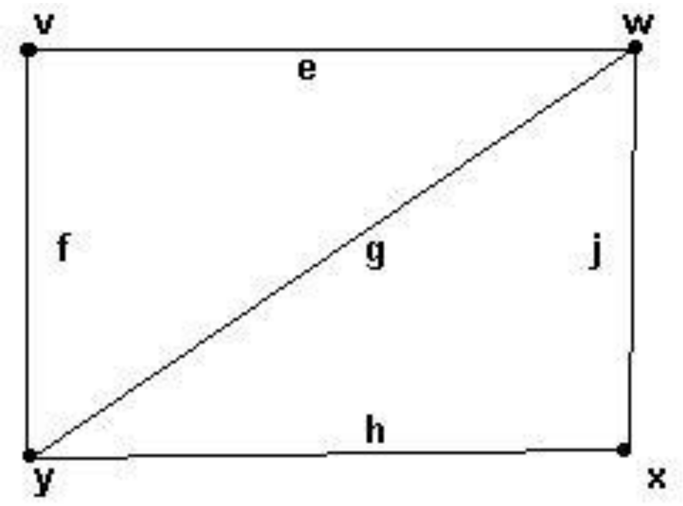
- **Adjacency matrix**
  - **Rows and columns are labeled with ordered vertices**
  - **write a 1 if there is an edge between the row vertex and the column vertex**
  - **and 0 if no edge exists between them**

	v	w	x	y
v	0	1	0	1
w	1	0	1	1
x	0	1	0	1
y	1	1	1	0



Incidence matrix

- **Incidence matrix**
  - **Label rows with vertices**
  - **Label columns with edges**
  - **1 if an edge is incident to a vertex, 0 otherwise**

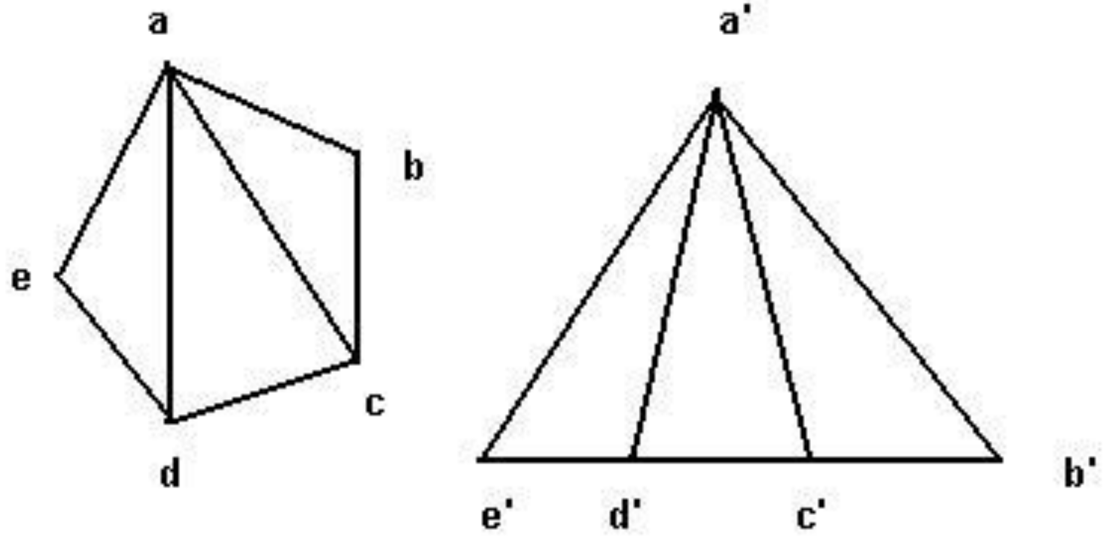


	e	f	g	h	j
v	1	1	0	0	0
w	1	0	1	0	1
x	0	0	0	1	1
y	0	1	1	1	0

Isomorphic graphs

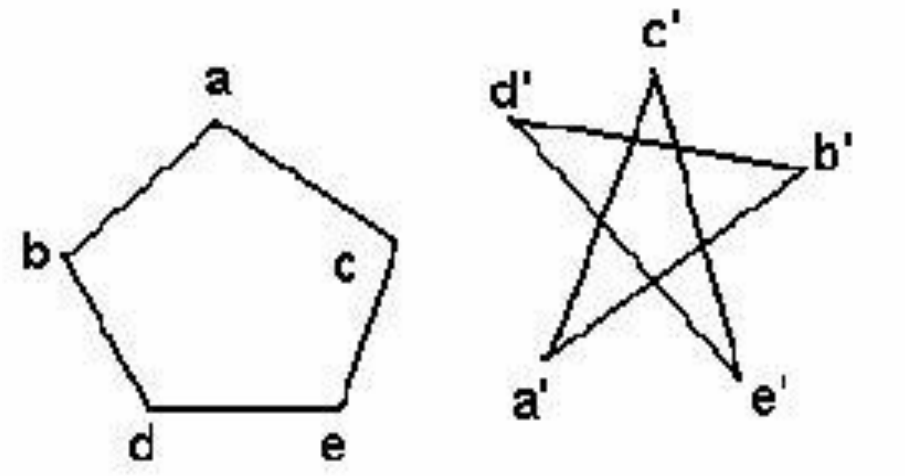
G_1 and G_2 are ***isomorphic***

- if there exist one-to-one onto functions f: $V(G_1) \to V(G_2)$ and g: $E(G_1) \to E(G_2)$ such that
- an edge e is adjacent to vertices v, w in G_1 if and only if g(e) is adjacent to f(v) and f(w) in G_2



Isomorphism and adjacency matrices

- **Two graphs are isomorphic if and only if after reordering the vertices their adjacency matrices are the same**



	a	b	c	d	e
a	0	1	1	0	0
b	1	0	0	1	0
c	1	0	0	0	1
d	0	1	0	0	1
e	0	0	1	1	0